Bose-Einstein Condensate Dark Matter Phase Transition from U(1) Symmetry Breaking

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Starting with a scalar field in a thermal bath using the one loop quantum corrections potential, we rewrite the Klein-Gordon equation in its thermodynamical representation and study the phase transition of this scalar field due to the $U(1)$ symmetry breaking. We find the corresponding thermodynamic and viscosity expressions and propose that these terms can be measured in the laboratory in order to know if the phase transition of a Bose-Einstein Condensate can be explained in terms of quantum field theory in a simple and elegant way.

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Introduction.- Symmetry breaking (SB) is one of the most essential concepts in particle theory and has been extensively used in the study of the behavior of particle interactions in many theories [1, 2].

Phase transitions are changes of state, related with changes of symmetries of the system. The analisys of SB mechanisms have turn out to be very helpful in the study of phenomena associated to phase transitions in almost all areas of physics. Bose-Einstein Condensation (BEC) is one topic of interest that uses in an extensive way the SB mechanisms [3]. The increasing studies in the correspondence between SB in condensed matter physics and cosmology are one of the motivations of this research [4]. Although phase transitions happen in condensed matter systems and the Universe, their status is not similar in both cases. For condensed matter, more transitions are known, for which our knowledge is better [5–7]. Nevertheless, there have been very few experiments connected to them in other directions likecosmology. The fact that condensed matter might permit for experiments that open the possibility of examining methods applied or used in cosmology and other areas of physics is one the most interesting aspects of such research.

The similarity between transitions in the Universe and condensed matter is exciting, but for them to be concluding we have the need of studying how transitions take place in condensed matter, so that we can see if the same assumptions about the Universe take us to understand it better. This might be done by looking for characteristics of transitions from which we can understand the nature of SB. The theoretical expectation that at high temperatures phase transitions might be possible in our Universe, associated with the breaking of symmetries, is of special concern for cosmology. The cosmic field scenarios follows the concept from particle physics and condensed matter physics that as soon as the temperature of a system decreases, then a field can develop a nearly definite classical value that can fix nearly all of its physical properties. These field energy concentrations can then maybe gravitationally deviate an initially uniform matter distribution, possibly seeding structure formation or giving other interesting results for the evolution of the Universe [8, 9].

The results from finite temperature quantum field theory raise important challenges about their possible manifestation in condensed matter systems, which can be drawn by non-relativistic or relativistic scalar field (SF) theories in the framework of the Ginzburg-Landau theory of condensed systems. Here, our aim is to study a relativistic model made up of a real SF together with the possibility that this SF dark matter might undergo a phase transition as the temperature of the Universe is lowered. To analyze this SB we take a model having an U(1) symmetry, considering the easiest case of a double-well interacting potential for a real SF $\Phi(\vec{x}, t)$ that goes as

$$V = \frac{1}{2}\frac{m^2c^2}{\hbar^2}\Phi^2 - \frac{\lambda}{4\hbar^2c^2}\Phi^4. \tag{1}$$

Here one important idea, to which we shall refer, is that of identifying the order parameter Ψ which characterizes the phase transition with the value of the real scalar quantum field Φ.

Theoretical framework.- In a expanding universe, the SF is governed by the Klein-Gordon (KG) equation given by

$$\Box\Phi + V_{,\Phi\Phi}\,\Phi + 2V_{,\Phi}\,\phi = 0, \tag{2}$$

where we have added an external interaction potential ϕ and the D'Alambertian operator which for an expanding Universe is defined as

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{3H}{c^2}\frac{\partial}{\partial t} - \frac{1}{a^2}\nabla^2.$$

being a the scale factor and $H = \dot{a}/a$ the Hubble parameter.

Here we are interested in the properties and behavior of the SF in a thermal bath, and in the interactions with other particles. In that case, the SF potential (1)

extended to one loop is [10, 11],

$$V=\frac{1}{2}\hat{m}^2\Phi^2-\frac{\hat{\lambda}}{4}\Phi^4-\frac{\hat{\lambda}}{8}k_B^2T^2\Phi^2+\frac{\pi^2k_B^4}{90\hbar^2c^2}T^4 \qquad (3)$$

where $\hat{\lambda}=\lambda/(\hbar^2c^2)$, $\hat{m}^2=m^2c^2/\hbar^2$, k_B is Boltzmann's constant and T is the temperature of the thermal bath, this result includes both quantum and thermal contributions. When the temperature T is high enough, the minimum of the potential (3) is $\Phi=0$. At this point, the SF density is equal to $\rho_\Phi=1/2\dot{\Phi}^2+V(\Phi)\sim T_0T^4$ and it behaves as a radiation fluid. Now, we suppose that the temperature is sufficiently small so that the interaction between the SF and the rest of matter decouples. After this moment, the term T_0T^4 is not longer important, thus, for sufficiently low T the radiation term can be dropped out.

The critical temperature where the minimum of the potential $\Phi=0$ becomes a maximum is

$$k_BT_c=\frac{2\hat{m}}{\sqrt{\lambda}}.$$

Now for the SF we perform the transformation

$$\kappa\Phi=\frac{1}{\sqrt{2}}\left(\Psi\,e^{-i\hat{m}ct}+\Psi^*\,e^{i\hat{m}ct}\right),$$

where κ is the scale of the system to be determined by the experiment and $\hat{m}$ is the SF mass in natural units, from now on we will take $c=1$.

In terms of function Ψ, the KG equation (2) reads,

$$\begin{aligned}-\,i\,\hbar(\dot{\Psi}+\frac{3}{2}H\Psi)\;+\;\frac{\hbar^2}{2\,m}(\Box\Psi-9\lambda|\Psi|^2\Psi)&\\ +\;\mu\Psi\phi-\lambda\frac{\hbar^2}{8m}k_B^2T^2\Psi=0,& \quad (4)\end{aligned}$$

where we have used the notation $\dot{}=\partial/\partial t$, $|\Psi|^2=\Psi\Psi^*=\rho$, and $\mu=m-\lambda\hbar^2k_B^2T^2/4m$. (4) is KG's equation in terms of the function Ψ and temperature T. From (4) we can notice that in a non-expanding Universe, i.e., $H=0$, when $T=0$ and in the non-relativistic limit, $\Box\to-\nabla^2$, eq. (4) becomes the Schrödinger equation with an extra term $|\Psi|^2$. In this limit this is the Gross-Pitaevskii equation for Bose-Einstein Condensates (BEC), i.e, an approximate equation for the mean-field order parameter. The static limit of equation (4) is the Ginzburg-Landau equation [12].

In what follows we transform the relativistic Gross-Pitaevskii like equation (4) into its analogous hydrodynamical version [13, 14]. For this, first we expand the function Ψ in its corresponding norm $\hat{\rho}$ and a phase S as

$$\Psi=\sqrt{\hat{\rho}}\,e^{iS}. \qquad (5)$$

Here we will interpret $\hat{\rho}=\rho/M_T$ as the rate between the number density of particles in the condensed state, $\rho=mn_0=mN_0/a^3$, being N_0 the number of particles in condensed state and M_T the total mass of the particles in the system, $S=S(\vec{x},t)$ will be a function related to the velocity field, as we will see later on, being both of them functions of time and position.

So, from this interpretation we have that when the KG equation oscillates around the $\Phi=0$ minimum, the number of particles in the ground state is zero, $\hat{\rho}=0=\rho$. Below the critical temperature T_c close to the second minimum, $\Phi^2_{min}=k_B^2(T_c^2-T^2)/4$, the density oscillates around $\hat{\rho}=\kappa^2k_B^2(T_c^2-T^2)/4$.

Then, from (4) and (5), separating real and imaginary parts we have

$$\dot{\hat{\rho}}+3H\hat{\rho}-\frac{\hbar}{m}\hat{\rho}\Box S+\frac{\hbar}{ma^2}\nabla S\nabla\hat{\rho}-\frac{\hbar}{m}\dot{\hat{\rho}}\dot{S}=0,$$

and

$$\begin{aligned}\frac{\hbar}{m}\dot{S}+\frac{\hbar^2}{2m^2a^2}(\nabla S)^2\;+\;w\hat{\rho}+\frac{\mu}{m}\phi-\lambda\frac{\hbar^2}{8m^2}k_B^2T^2&\\ +\;\frac{\hbar^2}{2m^2}\left(\frac{\Box\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}\right)-\frac{\hbar^2}{2m^2}(\dot{S})^2=0,&\end{aligned} \qquad (6)$$

where $w=-9\hbar^2\lambda/2m^2$. Now, taking the gradient of (6) then dividing by a and using the definition

$$\mathbf{v}\equiv\frac{\hbar}{ma}\nabla S$$

we obtain,

$$\dot{\hat{\rho}}+\frac{1}{a}\nabla\cdot(\hat{\rho}\mathbf{v})\;+\;3H\hat{\rho}\left(1-\frac{\hbar}{m}\dot{S}\right)-\frac{\hbar}{m}(\hat{\rho}\dot{S})^{\dot{}}=0, \qquad (7a)$$

$$\begin{aligned}\hat{\rho}\dot{\mathbf{v}}\left(1-\frac{\hbar}{m}\dot{S}\right)\;+\;\hat{\rho}\frac{1}{a}(\mathbf{v}\cdot\nabla)\mathbf{v}&\\ =\;\hat{\rho}\mathbf{F}_\phi-\frac{1}{a}\nabla p+\hat{\rho}\mathbf{F}_Q-\frac{1}{a}\hat{\rho}\nabla\sigma,&\end{aligned} \qquad (7b)$$

where $\mathbf{F}_\phi=-\nabla(\mu\phi)/ma$ is the force associated to the external potential ϕ, $p=w\hat{\rho}^2/2$ can be seen as the pressure of the SF gas, $\mathbf{F}_Q=-1/a\nabla U_Q$ is the quantum force associated to the quantum potential [15],

$$U_Q=-\frac{\hbar^2}{2m^2a^2}\left(\frac{\nabla^2\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}\right),$$

and $\hat{\rho}\nabla\sigma/a$ is defined as

$$\begin{aligned}\frac{1}{a}\hat{\rho}\nabla\sigma\;=\;H\mathbf{v}\hat{\rho}\left(1-\frac{\hbar}{m}\dot{S}\right)-\frac{1}{4}\frac{\lambda}{m^2a}k_B^2\hat{\rho}T\nabla T&\\ +\;\frac{1}{a}\zeta\nabla(\ln\hat{\rho})^{\dot{}}+\frac{\hbar^2\hat{\rho}}{4m^2a}\nabla\left(\frac{\ddot{\hat{\rho}}}{\hat{\rho}}\right),&\end{aligned} \qquad (8)$$

where the coefficient ζ is given by

$$\zeta = \frac{\hbar^2}{4m^2}\left[\frac{1}{a}\nabla\cdot(\hat\rho\mathbf{v}) + 6H\hat\rho - \frac{\hbar}{m}[3H\hat\rho\dot S + (\hat\rho\dot S)^{\dot{}}]\right],$$

and the term $\nabla(\ln\hat\rho)^{\dot{}}$ can be written as

$$\begin{aligned}\nabla(\ln\hat\rho)^{\dot{}} = &- \frac{1}{a}\nabla(\nabla\cdot\mathbf{v}) - \frac{1}{a}\nabla[(\nabla\ln\hat\rho)\cdot\mathbf{v}]\\ &+ 3H\frac{\hbar}{m}\nabla\dot S + \frac{\hbar}{m}\nabla\frac{1}{\hat\rho}(\hat\rho\dot S)^{\dot{}}\end{aligned}$$

System (7) is completely equivalent to equation (4).

Observe that from the uncertainty relation $mc^2\Delta t \sim \hbar$, we have that $\hbar/(mc^2)\dot S$ will only be important when the system is relativistic. In the non-relativistic limit, system (7) reads

$$\dot{\hat\rho} + \frac{1}{a}\nabla\cdot(\hat\rho\mathbf{v}) + 3H\hat\rho = 0, \tag{9a}$$

$$\hat\rho\dot{\mathbf{v}} + \hat\rho\frac{1}{a}(\mathbf{v}\cdot\nabla)\mathbf{v} = \hat\rho\mathbf{F}_\phi - \frac{1}{a}\nabla p + \hat\rho\mathbf{F}_Q - \frac{1}{a}\hat\rho\nabla\sigma, \tag{9b}$$

where the $\nabla(\ln\hat\rho)^{\dot{}}$ term now reads

$$\nabla(\ln\hat\rho)^{\dot{}} = -\frac{1}{a}\nabla(\nabla\cdot\mathbf{v}) - \frac{1}{a}\nabla[(\nabla\ln\hat\rho)\cdot\mathbf{v}].$$

Thus

$$\begin{aligned}\frac{1}{a}\hat\rho\nabla\sigma = H\hat\rho\mathbf{v} &- \frac{1}{4}\frac{\lambda}{m^2 a}k_B^2\hat\rho T\nabla T\\ &- \zeta\left[\frac{1}{a^2}\nabla(\nabla\cdot\mathbf{v}) + \frac{1}{a^2}\nabla[(\nabla\ln\hat\rho)\cdot\mathbf{v}]\right],\end{aligned}$$

where now we have

$$\zeta = \frac{\hbar^2}{4m^2}\left[\frac{1}{a}\nabla\cdot(\hat\rho\mathbf{v}) + 6H\hat\rho\right],$$

and for the non-relativistic limit we have neglected the gradient of the term $\frac{\hbar^2}{4m^2c^2}\left(\frac{\ddot{\hat\rho}}{\hat\rho}\right)$. Here we interpret the function $\nabla\sigma$ as a viscosity expression, it contains terms which are gradients of the temperature, of the divergence of the velocity and of the density, plus a term due to the expansion of the universe.

In what follows we will derive the thermodynamic equations from the KG equation. For doing so, we multiply (9b) by $\mathbf{v}$ and divide it by $\hat\rho$, with this we obtain,

$$\begin{aligned}(\frac{1}{2}\mathbf{v}^2)^{\dot{}} &+ \mathbf{v}^2\hat\nabla\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{F}_\phi + \frac{1}{\hat\rho}\mathbf{v}\cdot\hat\nabla p\\ &+ \frac{\hbar^2}{2m^2}\mathbf{v}\cdot\hat\nabla\left(\frac{\Box\sqrt{\hat\rho}}{\sqrt{\hat\rho}}\right) + H\mathbf{v}^2\\ &+ \frac{1}{4}\frac{\lambda}{m^2}k_B^2 T\mathbf{v}\cdot\hat\nabla T = 0\end{aligned}$$

with $\hat\nabla = \nabla/a$. In the same way, we can derive a conservation equation for the external potential $\hat\phi = \mu\phi$, given by the relationship

$$(\hat\rho\hat\phi)^{\dot{}} = \hat\rho\dot{\hat\phi} + \hat\phi\dot{\hat\rho} \tag{10}$$

Using the continuity equation (7a) into (10) we obtain,

$$(\hat\rho\hat\phi)^{\dot{}} + \hat\nabla\cdot(\hat\rho\mathbf{v}\hat\phi) + 3H\hat\rho\hat\phi = -\hat\rho\mathbf{v}\cdot\mathbf{F}_\phi + \hat\rho\dot{\hat\phi}.$$

Observe that we can generate a continuity equation even for U_Q in the same way as before, obtaining

$$(\hat\rho U_Q)^{\dot{}} + \hat\nabla\cdot(\hat\rho\mathbf{v}U_Q) + 3H\hat\rho U_Q = -\hat\rho\mathbf{v}\cdot\mathbf{F}_Q + \hat\rho\dot U_Q \tag{11}$$

Nevertheless, this procedure is not possible for σ because in general we do not know it explicitly, we only know its gradient, only in some cases it is possible to integrate it. However, the continuity equation for σ would be given by

$$(\hat\rho\sigma)^{\dot{}} + \nabla\cdot(\hat\rho\vec v\sigma) + 3H\hat\rho\sigma = \hat\rho\vec v\cdot\nabla\sigma + \hat\rho\dot\sigma.$$

Now observe that the quantum potential U_Q in (11) also fulfills the following relationship

$$\hat\rho\dot U_Q + \hat\nabla\cdot(\hat\rho\mathbf{v}_\rho) = 0,$$

which follows by direct calculation, and where we have defined the velocity density $\mathbf{v}_\rho$ by

$$\mathbf{v}_\rho = \frac{\hbar^2}{4m^2}(\nabla\ln\hat\rho)^{\dot{}},$$

which can be interpreted as the velocity flux of the potential U_Q. Using the continuity equation (9a), this last relation can be written as

$$(\hat\rho U_Q)^{\dot{}} + \hat\nabla\cdot(\hat\rho U_Q\mathbf{v} + \mathbf{J}_\rho) + 3H\hat\rho U_Q = 0 \tag{12}$$

where we have defined the quantum density flux

$$\mathbf{J}_\rho = \hat\rho\mathbf{v}_\rho.$$

Equation (12) is another expression for the continuity equation of the quantum potential U_Q.

On the other hand, the total energy of the system is given by

$$e = \frac{1}{2}\mathbf{v}^2 + \hat\phi + U_Q + \sigma + u \tag{13}$$

where u is the inner energy of the system. By definition, the total energy e must be conserved, that means that e must fulfill a continuity equations given by

$$(\hat\rho e)^{\dot{}} + \hat\nabla\cdot\mathbf{J}_e + 3H\hat\rho e + \hat\rho(\dot{\hat\phi} + \dot U_Q + \dot\sigma) + p\hat\nabla\cdot\mathbf{v} = 0, \tag{14}$$

being $\mathbf{J}_e$ the total energy current, given by the total energy flux and the heat flux, $\mathbf{J}_q$,

$$\mathbf{J}_e = \hat\rho e\mathbf{v} + \mathbf{J}_q + p\mathbf{v}.$$

We now substitute (13) into (14), in the non-relativistic limit we obtain,

$$(\hat{\rho}u)^{\cdot}+\hat{\nabla}\cdot(\hat{\rho}\mathbf{v}u+\mathbf{J}_q+p\mathbf{v}+\mathbf{J}_\rho)+\hat{\rho}(\dot{\dot{\phi}}+\dot{\sigma})+3H\hat{\rho}u=-p\nabla\cdot\mathbf{v} \quad (15)$$

In order to find the thermodynamical quantities of the system in equilibrium (when p is constant on a volume V), we restrict the system to the regime where the gravitational potential is nearly constant in time and where the viscosity coefficient fulfills the relation $\dot{\sigma}=0$. With this conditions at hand for (15) we have:

$$(\hat{\rho}u)^{\cdot}+\hat{\nabla}\cdot(\hat{\rho}\mathbf{v}u+\mathbf{J}_q+p\mathbf{v}+\mathbf{J}_\rho)+3H\hat{\rho}u=-p\nabla\cdot\mathbf{v} \quad (16)$$

Multiplying this last equation by a^3 and after integrating the resulting expression on a close region, we obtain

$$\frac{d}{dt}\int\hat{\rho}u\,a^3dV+\oint a^3(\mathbf{J}_q+p\mathbf{v})\cdot\mathbf{n}dS \;+\; \oint a^3\mathbf{J}_\rho\cdot\mathbf{n}dS \;=\; -p\frac{d}{dt}\int a^3dV.$$

Equation (16) is the continuity equation for the internal energy of the system and as usual, from here we can derive a generalized version of the first law of thermodynamics for the KG equation or a BEC, this reads

$$dU=\hat{\mathrm{d}}Q+\hat{\mathrm{d}}A_Q-pdV \quad (17)$$

where as always, $U=\int\hat{\rho}u\,a^3dV$ is the internal energy of the system, Q is the heat, and

$$\frac{\hat{\mathrm{d}}A_Q}{dt}=-\frac{\hbar^2}{4m^2}\oint a^3\hat{\rho}(\nabla\ln\hat{\rho})^{\cdot}\cdot\mathbf{n}dS=-\oint a^3\hat{\rho}\mathbf{v}_\rho\cdot\mathbf{n}dS$$

is the corresponding quantum heat flux due to the quantum nature of the KG equation. The second term on the right hand side of equation (17) would make the crucial difference between a classical and a quantum first law of thermodynamics.

Phase transition.- From hereafter we study the transition between the $\Phi=0$ state to the low energy one with $T<T_c$ close to the minimum $\Phi_{min}=k_B\sqrt{T_c^2-T^2}/2$.

We have already seen that during the epoch when $T>>T_c$ there are not scalar particles in the ground state. We will suppose that the scalar particles decouple from the rest of the matter at some moment, such that the total density here on remains constant. With the space-time expansion the temperature decreases, below the critical one $T<T_c$, close to the local minimum the density oscillates around the value $\hat{\rho}=k_B^2\kappa^2(T_c^2-T^2)/4$, being $k_B\kappa T_c\sim 1$. The function S in (5) has a simple expression, $S=s_0t$, with $s_0<<mc/\hbar$ in the non-relativistic case. This implies that the velocity $\mathbf{v}_0=0$, also, if there does not exist an external force in the system then, $\mathbf{F}_\phi=0$. If the temperature gradients are very small, in this limit the density is $\hat{\rho}=\hat{\rho}_0a^{-3}$, thus the temperature of the BEC decays as

$$T^2\propto T_c^2-\frac{4\hat{\rho}_0}{a^3}.$$

If there is no space-time expansion, the viscosity of the BEC might in fact contain the whole information of the phase transition.

Finally, to illustrate the previous exposition, we give the following example. Suppose that in the system there are only condensed and excited particles of the same specie. Thus $\hat{\rho}=N_0/(N_{ex}+N_0)$, where N_0 is the number of condensed particles and N_{ex} the number of exited particles. Combining the previous equations we obtain,

$$N_0=\frac{N}{k_B^2\kappa^2T_c^2}\left[1-\left(\frac{T}{T_c}\right)^2\right],$$

being $N=N_0+N_{ex}$. Observe that only a fraction $N/(k_B^2\kappa^2T_c^2)$ of the scalar particles reaches the ground state at $T=0$, this value can only be determined experimentally and fits the value of the scale κ. The main idea we want to point out here is that these phenomena might be equivalent for a BEC on earth as for the cosmos, and this might follow the previous equations exactly, so this function might be tested in the laboratory. If confirmed, the phase transition of a BEC can be explained using the quantum field theory in a very simple way. We expect that all of these results can be checked in an experiment, specially the behavior of the density expressions during the phase transition of a BEC.

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